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REGULAR THIN NEAR OCTAGONS HAVING LESS THAN 100 POINTS

SAEED A. SHAD, SIHAM J. AL-SAYYAD

and

SUHA A. WAZZAN

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Abstract

The main object of this paper is to study all regular thin near octagons with number of points less than or equal to 100. We find 10 necessary conditions for their existence. We prove the existence on non-existence of 23 feasible parameter sets of regular thin near octagons. We also find the related design or group if thin near octagon exists.

1. Introduction

The concept of a near $2n$-gon is due to Shult and Yanushka [12]. A *near $2n$-gon* is a linear incidence system $(\wp, \ell)$ of points and lines such that:

(i) Each line contains at least two points.

(ii) The distance between any two points is at most n.

(iii) For each point-line pair (p, L) there is a unique point on L nearest p.

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A near $2n$-gon has order (s, t) if each point lies on $1 + t$ lines and each line contains $1 + s$ points. A near $2n$-gon of order (s, t) is called *regular* with parameters $(s, t_2, t_3, \ldots, t_n = t)$ if whenever two points x and y are at distance $d \geq 1$, exactly $1 + t_d$ lines through y contain points at distance $d - 1$ from x. A regular near $2n$-gon is called *thin* if each line has exactly two points. If $t_2 = t_3 = \cdots = t_{n-1} = 0$, then the near $2n$-gon is a *generalized* $2n$-*gon* [9] of order (s, t). (Note that $t_0 = -1$, $t_1 = 0$, and $t_d \geq t_{d-1}$, $1 \leq d \leq n$.)

The main object of this paper is to study all possible regular thin near octagons having parameters $(s, t_2, t_3, t_4 = t)$ with $s = 1$ and $|\wp| \leq 100$.

2. Definitions and Known Results

Definition 2.1. A graph $\mathcal{G} = (\wp, \ell)$ is called *strongly regular* with parameters n, k, λ, μ such that $n = |\wp|$ and

(i) Each point is collinear with k other points.

(ii) Given two collinear points of $\wp$, there are λ points collinear to both of them.

(iii) Given two non-collinear points of $\wp$, there are μ points collinear to both of them.

Definition 2.2 [6, 9]. A generalized $2n$-gon $(n \geq 2)$ is a linear incidence system $(\wp, \ell)$ such that

(i) $d(x, y) \leq n$ for all $x, y \in \wp$.

(ii) Given $x \in \wp$, there exists $y \in \wp$ such that $d(x, y) = n$.

(iii) If $d(x, y) = m < n$, then there is exactly one path of length m from x to y.

A generalized $2n$-gon is called *regular* of order (s, t), with $s \geq 1$, $t \geq 1$ if each line contains $1 + s$ points and each point lies on $1 + t$ lines.

Definition 2.3. A generalized $2n$-gon is called

(1) a *generalized quadrangle* if $n = 2$.

(2) a *generalized hexagon* if $n = 3$.

(3) a *generalized octagon* if $n = 4$.

Definition 2.4. If a regular generalized $2n$-gon has order (s, t), then a regular generalized $2n$-gon having order (t, s) is called the *dual* of first one.

Theorem 2.5. *If a regular generalized $2n$-gon of order (s, t) exists, then its dual also exists.*

Theorem 2.6. *Let $(\wp, \ell)$ be a regular generalized $2n$-gon of order (s, t). Then for any point $x \in \wp$,*

$$|\Delta_d(x)| = s^d t^{d-1}(1 + t), \; 1 \le d < n$$

and

$$|\Delta_n(x)| = s^n t^{n-1}.$$

Lemma 2.7. *If $(\wp, \ell)$ is a regular generalized hexagon of order (s, t), then for any point $x \in \wp$,*

(i) $|\Delta_1(x)| = s(1 + t)$,

(ii) $|\Delta_2(x)| = s^2 t(1 + t)$,

(iii) $|\Delta_3(x)| = s^3 t^2$,

(iv) $|\wp| = (1 + s)(1 + st + s^2 t^2)$,

(v) $|\ell| = (1 + t)(1 + st + s^2 t^2)$.

Example 2.8 [10]. Let $G = \text{Alt}\ (6)$, the alternating group on 6 letters $\{1, 2, 3, 4, 5, 6\}$. Define a system of points and lines as follows:

$$\wp = \{t \mid t \in Inv(G), \text{ the single class of involutions in } G\},$$

two points $t_1, t_2 \in \wp$ are collinear if and only if $t_1 t_2 = t_2 t_1$. Then the incidence structure $(\wp, \ell)$ is a generalized octagon of order $(2, 1)$.

Lemma 2.9. *Any generalized 2n-gon is near 2n-gon.*

Theorem 2.10. *Let* $(\wp, \ell)$ *be a regular near 2n-gon with parameters* $(s, t_2, t_3, \ldots, t_n = t)$. *Then for any point* $x \in \wp$,

$$| \Delta_1(x) | = s(1 + t),$$

$$| \Delta_d(x) | = \frac{s^d (Ht) t(t - t_2)(t - t_3) \cdots (t - t_{d-1})}{(1 + t_2)(1 + t_3) \cdots (1 + t_d)},$$

for $2 \leq d \leq n$.

Definition 2.11. A *balanced incomplete block design* (BIBD) with parameters (v, b, r, k, λ) is an arrangement of v distinct objects (elements) into b blocks such that

(1) each block has exactly k distinct objects,

(2) each object occurs in exactly r different blocks,

(3) each pair of objects lies in exactly λ blocks.

A BIBD with parameters (v, b, r, k, λ) is called *symmetric* if $v = b$ (and so, of course $k = r$). For a symmetric BIBD, the parameters are (v, k, λ).

Definition 2.12. A *resolvable design* is a BIBD having parameters (v, b, r, k, λ) with $b = nr$ if it is possible to partition the set of b blocks into r subsets of n blocks each, so that each object occurs exactly once among the blocks of a given subset (i.e., each subset contains a complete replication).

Definition 2.13. An *affine resolvable design* is a resolvable design for which any two blocks coming from different subsets intersect in the same number of objects.

Theorem 2.14 [1]. *If a* (v, b, r, k, λ) *design is resolvable, and* $b = v + r - 1$, *then the design is affine resolvable and any two blocks* *coming from different subsets intersect in* $\frac{k^2}{v}$ *objects.*

Definition 2.15. *A steiner system* $S(l, m, n)$ is a collection of m-element subsets of an n-element set B such that every l-element subset of B lies in exactly one of the m-element subsets.

Example 2.16. A BIBD with parameters $(v, b, r, k, \lambda) = (9, 12, 4, 3, 1)$ is an example of steiner system $S(2, 3, 9)$.

Definition 2.17. Let $\mathcal{G} = (\wp, \ell)$ be the graph of a regular near $2n$-gon. We may define an *adjacency matrix* A for this graph. The point set $\wp$ is totally ordered and the rows and columns of A are indexed by this ordering. A contains the entry a_{st} in the s-th row and t-th column where

$$a_{st} = \begin{cases} 0, & \text{if } s\text{-th and } t\text{-th points are not collinear in } \mathcal{G}. \\ 1, & \text{if } s\text{-th and } t\text{-th points are collinear in } \mathcal{G}. \end{cases}$$

Theorem 2.18. *Let A be the adjacency matrix of a regular thin near octagon $(\wp, \ell)$ with parameters $(1, t_2, t_3, t)$. Then*

(a) *the eigenvalues of A satisfy the equation:*

$$(x - 1 - t)(x)(x + 1 + t)(x^2 + t_2 + t_3 + t_2t_3 - tt_2 - 2t) = 0.$$

(b) $k = 1 + t$ *is an eigenvalue of A with multiplicity one.*

(c) *the matrix A has five distinct eigenvalues* k, u_1, u_2, u_3, u_4, *where*

$$k = 1 + t,$$

$$u_1 = -(1 + t),$$

$$u_2 = \sqrt{d}, \text{ where } d = 2t + tt_2 - t_2 - t_3 - t_2t_3,$$

$$u_3 = -\sqrt{d},$$

$$u_4 = 0.$$

(d) *if* f_1, f_2, f_3, f_4 *are the multiplicities of the eigenvalues* $u_1, u_2, u_3,$

u_4 *respectively, then*

$$f_1 = 1,$$

$$f_2 = \frac{n(1+t) - 2(1+t)^2}{2d} = f_3, \text{ where } n = |\wp|,$$

$$f_4 = n - 2f_2 - 2.$$

Corollary 2.19. *The multiplicities* f_2, f_3 *of the eigenvalues* u_2, u_3 *in above Theorem* 2.18 *may be written as*

$$f_2 = \frac{t(t-t_2)(1+t)^2}{(1+t_2)(1+t_3)d} = f_3, \text{ where } d = tt_2 + 2t - t_2 - t_3 - t_2t_3.$$

Definition 2.20 [4]. For $\lambda > 1$, a *partial* λ*-geometry* with nexus e is an incidence structure with v points and v blocks such that

(i) each two points are joined by 0 or λ blocks,

(ii) each two blocks have 0 or λ points in common,

(iii) each point lies in k blocks and each block has k points,

(iv) if p is any point and B is any block such that $p \notin B$, then there are exactly e blocks C with p in C such that $B \cap C$ is not empty.

Definition 2.21. If $\mathcal{G}$ is a partial λ-geometry with $\lambda > 1$ and $k > e$, then $\mathcal{G}$ is called a *proper partial* λ*-geometry*. In this case, (v, λ, e, k) are called the *parameters* of proper partial λ-geometry.

Lemma 2.22 [2]. *A proper partial* λ*-geometry* $\mathcal{G}$ *with parameters* (v, λ, e, k) *is nothing but a regular thin near octagon* $(\wp, \ell)$, *where* $\wp = \wp_1 \cup \wp_2$ *and*

$\wp_1$ = *the set of points of* $\mathcal{G}$,

$\wp_2$ = *the set of blocks of* $\mathcal{G}$,

and lines in $(\wp, \ell)$ *are defined to be the incident point-block pair* $(p, B) \cdot (\wp, \ell)$ *has parameters*

$$(s, t_2, t_3, t) = (1, \lambda - 1, e - 1, k - 1) \text{ and } |\wp| = 2v.$$

Definition 2.23. A near $2n$-gon $(\wp, \ell)$ is called *non-degenerate* if there exist $x, y \in \wp$ such that $d(x, y) = n$.

Lemma 2.24. *If $(\wp, \ell)$ is a non-degenerate regular thin near octagon with parameters $(1, t_2, t_3, t)$, then $t > t_3$.*

Theorem 2.25 [4]. *Suppose there exists a regular thin near octagon $(\wp, \ell)$ with parameters $(1, t_2, t_3, t)$, where $t_2 \geq 1$, then there exists a strongly regular graph having parameters (m, a, c, d) such that*

$$m = \frac{|\wp|}{2} = \frac{t(1+t)(t-t_2)}{(1+t_2)(1+t_3)} + (1+t),$$

$$a = \frac{t(1+t)}{1+t_2}$$

$$c = (t - t_3 - 1) + \frac{(1+t)t_3}{1+t_2}$$

$$d = \frac{(1+t)(1+t_3)}{1+t_2}.$$

Corollary 2.26. *If $(\wp, \ell)$ is a regular thin near octagon with parameters $(1, t_2, t_3, t)$, then $(1 + t_2)$ divides $(1 + t)$.*

Theorem 2.27 [4]. *Let $(\wp, \ell)$ be a regular thin near octagon with parameters $(1, t_2, t_3, t)$.*

(i) *If $t_2 = 1$, then $t_3 \geq 2$.*

(ii) *If t_2 is an even integer ≥ 2, and $t > 1 + t_3$, then $t_3 > 2t_2 + 1$.*

(iii) *If t_2 is an odd integer ≥ 3, and $t > 1 + t_3$, then $t_3 > 2t_2 + 2$.*

Theorem 2.28 [11]. *If $(\wp, \ell)$ is a regular thin near octagon with parameters $(1, t_2, t_3, t) = (1, t_2, 2t_2, 2t_2 + 1)$, $t_2 \neq 0$, then t_2 must be an odd integer.*

Definition 2.29. Let $\mathcal{G}$ be a linear incidence system of points and blocks. Define $B \parallel G$ for blocks B, G of $\mathcal{G}$ to mean either $B = G$ or

$[B, G] = 0$, where $[p, q]$ denotes the number of blocks that contain the point set $\{p, q\}$ and $[G, H]$ denotes the number of points common to the block set $\{G, H\}$.

Definition 2.30. A *parallelism* on an incidence system of points and blocks is an equivalence relation on the set of blocks such that each equivalence class, called a *parallel class*, partitions the point set.

Definition 2.31 ([5, 8]). Let $\mathcal{G}$ be an incidence system of points and blocks. Then $\mathcal{G}$ is called an (s, r, μ)*-net* if

(i) $\|$ is a parallelism,

(ii) $G \nparallel H \Rightarrow [G, H] = \mu$ for blocks G, H of $\mathcal{G}$,

(iii) there is at least one point, some parallel class has $s \geq 2$ blocks, and there are $r \geq 3$ parallel classes.

$\mathcal{G}$ is called an *affine resolvable partial plane* if, in addition, there exists an integer λ such that

(iv) $[p, q] = 0$ or λ, whenever $p \neq q$.

Theorem 2.32. *Let $\mathcal{G}$ be an (s, r, μ)-net. Then $\mathcal{G}$ has*

(i) $v = s^2\mu$ *points.*

(ii) $b = sr$ *blocks.*

(iii) *s blocks in every parallel class.*

(iv) $k = su$ *points per block.*

(v) *If $\mathcal{G}$ is affine resolvable partial plane (ARPP), then*

$$(\lambda - 1)(s\mu - 1) = (r - 1)(\mu - 1).$$

Definition 2.33. Let $\mathcal{G}$ be an (s, r, μ)-net. Then $\mathcal{G}$ is called *quasi-symmetric* if $\lambda = \mu$ and $\mathcal{G}$ is called *symmetric* if $r = k$.

Definition 2.34. A *Hadamard Matrix of order m* is an $m \times m$ matrix H of $+1$'s and -1's such that $HH^t = mI$, where H^t is the transpose matrix of H and I is the identity matrix.

Theorem 2.35 [11]. *For $u > 1$, a symmetric $(2, 2u, u)$-net $\mathcal{G}$ exists if and only if there exists a Hadamard matrix H of order $2u$.*

Corollary 2.36. *For $u \geq 1$, a regular thin near octagon with parameters $(1, t_2, t_3, t) = (1, 2u-1, 4u-2, 4u-1)$ exists if and only if there exists a symmetric net $(s, r, \mu) = (2, 4u, 2u)$.*

Theorem 2.37 [5]. *Let p be a prime and α, β be non-negative integers with $\beta \geq \max(1, \alpha)$. Then there exists a symmetric (s, r, μ)-net with*

$$s = p, \quad r = 2^{\alpha} p^{\beta}, \quad \mu = 2^{\alpha} p^{\beta-1}, \quad \text{unless } r = 2.$$

Theorem 2.38 [5]. *Let p be a prime and i, j be integers with $i \geq 1$, $j \geq 0$. Then there exists a symmetric (s, r, μ)-net with*

$$s = p^{i}, \quad r = p^{i+j}, \quad \mu = p^{j}.$$

Theorem 2.39 [5]. *A regular thin near octagon with parameters $(1, t_2, t_3, t) = (1, k-1, mk-2, mk-1)$ exists if and only if there exists a symmetric net $(s, r, \mu) = (m, mk, k)$, where $k \geq 1$ and $mk \geq 3$.*

3. Feasible Parameter Sets for Regular Thin Near Octagons

Theorem 3.1 [3]. *Let $(\wp, \ell)$ be a regular near octagon with parameters (s, t_2, t_3, t). Then one of the following holds:*

(i) $s = 1$; *or*

(ii) $t_2 = 0$; *or*

(iii) $t_2 = 1$; *or*

(iv) $t_3 = t_2(t_2 + 1)$ *and* $t_4 = t_2(t_3 + 1)$.

This theorem shows the nonexistence of most regular near octagons.

Theorem 3.2. *Suppose a regular near octagon $(\wp, \ell)$ with parameters $(1, t_2, t_3, t)$ exists. Then the parameters must satisfy the following ten necessary conditions:*

(1) $|\Delta_2(x)| = \dfrac{t(1+t)}{1+t_2} \in \mathbb{N}$, *where* $x \in \wp$ (*Theorem* 2.10).

(2) $|\Delta_3(x)| = \dfrac{t(1+t)(t-t_2)}{(1+t_2)(1+t_3)} \in \mathbb{N}$, *where* $x \in \wp$ (*Theorem* 2.10).

(3) $|\Delta_4(x)| = \dfrac{t(t-t_2)(t-t_3)}{(1+t_2)(1+t_3)} \in \mathbb{N}$, *where* $x \in \wp$ (*Theorem* 2.10).

(4) $f_2 = f_3 = \dfrac{t(t-t_2)(1+t)^2}{(1+t_2)(1+t_3)d} \in \mathbb{N}$, *where* $d = tt_2 + 2t - t_2 - t_3 - t_2t_3$ (*Corollary* 2.19).

(5) $t > t_3$ (*Lemma* 2.24).

(6) $(1+t_2)$ *divides* $(1+t)$ (*Corollary* 2.26).

(7) *If* $t_2 = 1$, *then* $t_3 \geq 2$ (*Theorem* 2.27).

(8) *If* t_2 *is an even integer* ≥ 2, *and* $t > 1 + t_3$, *then* $t_3 > 2t_2 + 1$ (*Theorem* 2.27).

(9) *If* t_2 *is an odd integer* ≥ 3, *and* $t > 1 + t_3$, *then* $t_3 > 2t_2 + 2$ (*Theorem* 2.27).

(10) *If* $t_2 \neq 0$, $t_3 = 2t_2$, $t = 1 + t_3$, *then* t_2 *must be an odd integer* (*Theorem* 2.28).

Definition 3.3. A family of parameter sets $(1, t_2, t_3, t)$ of a regular thin near octagon is called *feasible* if the parameters t_2, t_3, t satisfy all the necessary conditions listed in Theorem 3.2.

Theorem 3.4. *If* $t_2 \neq 0$, *and* $1 + t = (1+t_2)(1+t_3)$, *then* $(1, t_2, t_3, t)$ *is a feasible family of parameter sets.*

Proof.

$$|\Delta_2(x)| = \frac{t(1+t)}{1+t_2} = \frac{t(1+t_2)(1+t_3)}{1+t_2} = t(1+t_3) \in \mathbb{N}.$$

$$|\Delta_3(x)| = \frac{t(t-t_2)(1+t)}{(1+t_2)(1+t_3)} = \frac{t(t-t_2)(1+t_2)(1+t_3)}{(1+t_2)(1+t_3)} = t(t-t_2) \in \mathbb{N}.$$

$$|\Delta_4(x)| = \frac{t(t-t_2)(t-t_3)}{(1+t_2)(1+t_3)} = \frac{tt_3(1+t_2)(1+t_3)t_2}{(1+t_2)(1+t_3)} = t_2t_3t \in \mathbb{N}.$$

$$f_2 = \frac{t(t-t_2)(1+t)^2}{(1+t_2)(1+t_3)d},$$

where

$$\begin{aligned} d &= t(t_2+2) - (t_2+t_3+t_2t_3) \\ &= t(1+t_2) + t - (t_2+t_3+t_2t_3) \\ &= t(1+t_2),\ t - t_2 = t_3(1+t_2). \end{aligned}$$

So

$$f_2 = \frac{tt_3(1+t_2)(1+t)(1+t_2)(1+t_3)}{(1+t_2)(1+t_3)(1+t_2)t} = t_3(1+t) \in \mathbb{N}.$$

$t_2 \neq 0$ implies $t > t_3$ and clearly $(1+t_2)$ divides $1+t$.

Corollary 3.5. *If* $(\wp, \ell)$ *is a regular thin near octagon with parameters* $(1, t_2, t_3, t)$ *and* $|\wp| \leq 100$, *then above Theorem* 3.4 *implies*

(i) $(1, t_2, t_3, t) = (1, 1, 2, 5)$.

(ii) $(1, t_2, t_3, t) = (1, 1, 3, 7)$.

Theorem 3.6. *If* $t - t_2 = (1+t_2)(1+t_3)$, *then* $(1, t_2, t_3, t)$ *is a feasible family of parameter sets.*

Proof. $t - t_2 = (1+t_2)(1+t_3)$ implies $1 + t = (1+t_2)(2+t_3)$,

$$|\Delta_2(x)| = \frac{t(1+t)}{1+t_2} = \frac{t(1+t_2)(2+t_3)}{1+t_2} = t(2+t_3) \in \mathbb{N}.$$

$$|\Delta_3(x)| = \frac{t(1+t)(t-t_2)}{(1+t_2)(1+t_3)} = \frac{t(1+t)(1+t_2)(1+t_3)}{(1+t_2)(1+t_3)} = t(1+t) \in \mathbb{N}.$$

$$|\Delta_4(x)| = \frac{t(t-t_2)(t-t_3)}{(1+t_2)(1+t_3)} = \frac{t(1+t_2)(1+t_3)(t-t_3)}{(1+t_2)(1+t_3)} = t(t-t_3) \in \mathbb{N}.$$

$$f_2 = \frac{t(t-t_2)(1+t)^2}{(1+t_2)(1+t_3)d},$$

where

$$\begin{aligned} d &= t(t_2 + 2) - (t_2 + t_3 + t_2t_3) \\ &= t(1 + t_2) + (t - t_2) - t_3(1 + t_2) \\ &= t(1 + t_2) + (1 + t_2)(1 + t_3) - t_3(1 + t_2) \\ &= (1 + t_2)(t + 1 + t_3 - t_3) \\ &= (1 + t_2)(1 + t). \end{aligned}$$

So $f_2 = \dfrac{t(1+t)(1+t_2)(2+t_3)}{(1+t_2)(1+t)} = t(2 + t_3) \in \mathbb{N}$.

Clearly, $(1 + t_2)$ divides $1 + t$ and $t > t_3$.

Corollary 3.7. *If* $(\wp, \ell)$ *is a regular thin near octagon with parameters* $(1, t_2, t_3, t)$ *and* $|\wp| \leq 100$, *then above Theorem* 3.6 *implies*

(i) $(1, t_2, t_3, t) = (1, 0, 0, 1)$,

(ii) $(1, t_2, t_3, t) = (1, 0, 1, 2)$,

(iii) $(1, t_2, t_3, t) = (1, 0, 2, 3)$,

(iv) $(1, t_2, t_3, t) = (1, 0, 3, 4)$,

(v) $(1, t_2, t_3, t) = (1, 0, 4, 5)$,

(vi) $(1, t_2, t_3, t) = (1, 0, 5, 6)$.

Theorem 3.8. *If* $(1 + t_2)$ *divides* $(t - t_2)$, *where* $t_2 \neq 0$; *then* $(1, t_2, t_3, t) = (1, t_2, t-1, t)$ *is a feasible family of parameter sets.*

Proof. Suppose $\dfrac{t - t_2}{1 + t_2} = a \in \mathbb{N}$, so $1 + t = (1 + t_2)(1 + a)$, $1 + t_3 = t$,

$$|\Delta_2(x)| = \frac{t(1 + t_2)}{1 + t_2} = (1 + a)t \in \mathbb{N}.$$

$$|\Delta_3(x)| = \frac{t(1 + t)(t - t_2)}{(1 + t_2)(1 + t_3)} = a(1 + t) \in \mathbb{N}.$$

$$|\Delta_4(x)| = \frac{t(t - t_2)(t - t_3)}{(1 + t_2)(1 + t_3)} = \frac{t - t_2}{1 + t_2} = a \in \mathbb{N}.$$

$$f_2 = \frac{t(t - t_2)(1 + t)^2}{(1 + t_2)(1 + t_3)d},$$

where

$$\begin{aligned} d &= t(t_2 + 2) - (t_2 + t_3 + t_2 t_3) \\ &= t(t_2 + 2) - t_2 - (t - 1) - t_2(t - 1) \\ &= 1 + t. \end{aligned}$$

So $f_2 = a(1 + t) \in \mathbb{N}$.

Clearly, $(1 + t_2)$ divides $(1 + t)$ and $t > t_3 = t - 1$.

Corollary 3.9. *If* $(\wp, \ell)$ *is a regular thin near octagon with parameters* $(1, t_2, t_3, t)$ *and* $|\wp| \leq 100$, *then above Theorem* 3.8 *and Theorem* 3.2 *imply*

(i) $(1, t_2, t_3, t) = (1, 1, 2, 3)$,

(ii) $(1, t_2, t_3, t) = (1, 1, 4, 5)$,

(iii) $(1, t_2, t_3, t) = (1, 1, 6, 7)$,

(iv) $(1, t_2, t_3, t) = (1, 1, 8, 9)$,

(v) $(1, t_2, t_3, t) = (1, 2, 7, 8)$,

(vi) $(1, t_2, t_3, t) = (1, 2, 10, 11)$,

(vii) $(1, t_2, t_3, t) = (1, 3, 6, 7)$,

(viii) $(1, t_2, t_3, t) = (1, 3, 10, 11)$,

(ix) $(1, t_2, t_3, t) = (1, 4, 13, 14)$,

(x) $(1, t_2, t_3, t) = (1, 5, 10, 11)$,

(xi) $(1, t_2, t_3, t) = (1, 7, 14, 15)$,

(xii) $(1, t_2, t_3, t) = (1, 9, 18, 19)$,

(xiii) $(1, t_2, t_3, t) = (1, 11, 22, 23)$.

Theorem 3.10. $(1, t_2, t_3, t) = (1, t_2, t_2 + t_2^2, t_2 + t_2^2 + t_2^3)$, *where* $t_2 \geq 1$, *is a feasible family of parameter sets.*

Proof.

$$1 + t = 1 + t_2 + t_2^2 + t_2^3 = (1 + t_2)(1 + t_2^2).$$

$$t - t_2 = t_2^2 + t_2^3 = t_2^2(1 + t_2),\ t_2(1 + t_3) = t.$$

$$|\Delta_2(x)| = \frac{t(1+t)}{1+t_2} = t(1 + t_2^2) \in \mathbb{N}.$$

$$|\Delta_3(x)| = \frac{t(1+t)(t-t_2)}{(1+t_2)(1+t_3)} = t_2^3(1 + t) \in \mathbb{N}.$$

$$|\Delta_4(x)| = \frac{t(t-t_2)(t-t_3)}{(1+t_2)(1+t_3)} = t_2^6 \in \mathbb{N}.$$

$$f_2 = \frac{t(t-t_2)(1+t)^2}{(1+t_2)(1+t_3)d},$$

where $d = t(t_2 + 2) - t_2 - t_3 - t_2t_3 = t_2^2(1 + t_2)^2$.

So $f_2 = t_2(1 + t_2^2)^2 \in \mathbb{N}$.

Clearly, $(1 + t_2)$ divides $(1 + t)$ and $t > t_3$.

Corollary 3.11. *If* $(\wp, \ell)$ *is a regular thin near octagon with parameters* $(1, t_2, t_3, t) = (1, t_2, t_2 + t_2^2, t_2 + t_2^2 + t_2^3)$ *and* $|\wp| \leq 100$, *then above Theorem* 3.10 *implies that*

$$(1, t_2, t_3, t) = (1, 1, 2, 3).$$

But this parameter set is already listed in Corollary 3.9.

Corollary 3.12. *If* $(\wp, \ell)$ *is a regular thin near octagon with parameters* $(1, t_2, t_3, t) = (1, 0, 0, t)$ *and* $|\wp| \leq 100$, *then Theorem* 2.6 *implies that*

(i) $(1, 0, 0, t) = (1, 0, 0, 1)$,

(ii) $(1, 0, 0, t) = (1, 0, 0, 2)$,

(iii) $(1, 0, 0, t) = (1, 0, 0, 3)$.

But first parameter set $(1, 0, 0, 1)$ *is already listed in Corollary* 3.7.

4. Existence or Non-existence of Regular Thin Near Octagons with Feasible Parameter Sets

4.1. Parameter sets of the form $(s, t_2, t_3, t) = (1, 0, 0, t)$

In this family of parameters $(s, t_2, t_3, t) = (1, 0, 0, t)$, we have only three feasible parameter sets with $|\wp| \leq 100$ (see Corollary 3.12).

4.1.1. The parameter set $(s, t_2, t_3, t) = (1, 0, 0, 1)$

In this case regular thin octagon $(\wp, \ell)$ exists and is the unique ordinary regular octagon with $|\wp| = 8 = |\ell|$.

4.1.2. The parameter set $(s, t_2, t_3, t) = (1, 0, 0, 2)$

We know from Example 2.8 that a regular generalized octagon of order $(2, 1)$ exists. We also know the following facts:

(1) A regular generalized $2n$-gon of order (s, t) is a regular near $2n$-gon with parameters: $(s, t_2, t_3, ..., t_{n-1}, t)$ such that $t_2 = t_3 = \cdots = t_{n-1} = 0$ (see Lemma 2.9).

(2) If a regular generalized $2n$-gon of order (s, t) exists, then its dual regular generalized octagon of order (t, s) also exists (see Theorem 2.5).

Using the above result (2), the existence of a regular generalized octagon of order $(2, 1)$ implies the existence of a regular generalized octagon of order $(1, 2)$. But then the result (1) above implies that a regular thin near octagon with parameters $(1, 0, 0, 2)$ exists.

4.1.3. The parameter set $(s, t_2, t_3, t) = (1, 0, 0, 3)$

The existence or non-existence of a regular thin near octagon $(\wp, \ell)$

having parameters (1, 0, 0, 3) is not yet determined. It is still an open question.

4.2. Parameter sets of the form $(s, t_2, t_3, t) = (1, 2u-1, 4u-2, 4u-1)$, $u \geq 1$

In this family of parameters $(s, t_2, t_3, t) = (1, 2u-1, 4u-2, 4u-1)$, where $u \geq 1$, we have six feasible parameter sets with $|\wp| \leq 100$ (see Corollary 3.9).

Theorem 4.2.1 [1]. *Let* $u \geq 1$. *If* $4u-1$ *is a prime power, then there exists an affine resolvable design with parameters* $(v, b, r, k, \lambda) = (4u, 8u-2, 4u-1, 2u, 2u-1)$.

Corollary 4.2.2. *There exists an affine resolvable design with the following parameters:*

(i) $(v, b, r, k, \lambda) = (4, 6, 3, 2, 1)$,

(ii) $(v, b, r, k, \lambda) = (8, 14, 7, 4, 3)$,

(iii) $(v, b, r, k, \lambda) = (12, 22, 11, 6, 5)$,

(iv) $(v, b, r, k, \lambda) = (20, 38, 19, 10, 9)$,

(v) $(v, b, r, k, \lambda) = (24, 46, 23, 12, 11)$.

Lemma 4.2.3 [7]. *There exists an affine resolvable design with parameters* $(v, b, r, k, \lambda) = (16, 30, 15, 8, 7)$.

Proof. Thirty blocks given by

$$(\infty, 0, 1, 2, 7, 9, 12, 13), (3, 4, 5, 6, 8, 10, 11, 14) \text{ modulo } 15$$

form an affine resolvable design with parameters (16, 30, 15, 8, 7).

Theorem 4.2.4 [11]. *A regular thin near octagon with parameters* $(s, t_2, t_3, t) = (1, 2u-1, 4u-2, 4u-1)$ *exists if and only if there exists an affine resolvable design having parameters* $(v, b, r, k, \lambda) = (4u, 8u-2, 4u-1, 2u, 2u-1)$, *where* $u \geq 1$.

Corollary 4.2.5. *Regular thin near octagon with the following parameters do exist*:

(i) $(1, t_2, t_3, t) = (1, 1, 2, 3)$,

(ii) $(1, t_2, t_3, t) = (1, 3, 6, 7)$,

(iii) $(1, t_2, t_3, t) = (1, 5, 10, 11)$,

(iv) $(1, t_2, t_3, t) = (1, 7, 14, 15)$,

(v) $(1, t_2, t_3, t) = (1, 9, 18, 19)$,

(vi) $(1, t_2, t_3, t) = (1, 11, 22, 23)$.

Proof. Using Corollary 4.2.2, Lemma 4.2.3, and Theorem 4.2.4, the required result is quite obvious.

Example 4.2.6. In this example, we explain how to construct a regular thin near octagon with parameters $(1, 7, 14, 15)$ from a given affine resolvable design D having parameters $(16, 30, 15, 8, 7)$.

Let $A = \{1, 2, ..., 16\}$ be the set of 16 points and $B = \{B_1, B_2, ..., B_{30}\}$ be the set of 30 blocks of the design D. Let $A' = \{1', 2', ..., 16'\}$ be an isomorphic copy of A with $A \cap A' = \varnothing$. Let $\wp = \{\infty\} \cup \{\infty'\} \cup A \cup A' \cup B$, where ∞ and ∞' are new points. We now construct $\mathcal{G} = (\wp, \ell)$ as follows:

(i) ∞ is joined to each point in A.

(ii) ∞' is joined to each point in A'.

(iii) each point $a \in A$ is joined to the 15 blocks containing a.

(iv) each point $a' \in A'$ is joined to the 15 blocks missing a.

Under this construction the graph $\mathcal{G} = (\wp, \ell)$ is a regular thin near octagon with parameters $(1, 7, 14, 15)$ and for any $x \in \wp$, we have

$$\begin{aligned} |\wp| &= 1 + |\Delta_1(x)| + |\Delta_2(x)| + |\Delta_3(x)| + |\Delta_4(x)| \\ &= 1 + 16 + 30 + 16 + 1 \\ &= 64. \end{aligned}$$

4.3. Parameter sets of the form $(s, t_2, t_3, t) = (1, 0, t-1, t)$, $t \geq 2$

In this family of parameters $(s, t_2, t_3, t) = (1, 0, t-1, t)$, where $t \geq 2$, we have five feasible parameter sets with $|\wp| \leq 100$ (see Corollary 3.7).

Theorem 4.3.1 [5]. *Let p be a prime number and* α, β *be non-negative integers with* $\beta \geq \max(1, \alpha)$. *Then there exists a symmetric* (s, r, μ)*-net with*

$$s = p,\ r = 2^{\alpha} p^{\beta},\ \ \mu = 2^{\alpha} p^{\beta-1},\ \ \textit{unless}\ \ r = 2.$$

Theorem 4.3.2 [5]. *Let p be a prime number and* i, j *be integers with* $i \geq 1$, $j \geq 0$. *Then there exists a symmetric* (s, r, μ)*-net with*

$$s = p^{i},\ \ r = p^{i+j},\ \ \mu = p^{j}.$$

Theorem 4.3.3 [5]. *A regular thin near octagon with parameters* $(1, t_2, t_3, t) = (1, k-1, mk-2, mk-1)$ *exists if and only if there exists a symmetric net* $(s, r, \mu) = (m, mk, k)$, *where* $k \geq 1$ *and* $mk \geq 3$.

Corollary 4.3.4. *Symmetric nets with the following parameters exist:*

(i) $(s, r, \mu) = (3, 3, 1)$,

(ii) $(s, r, \mu) = (4, 4, 1)$,

(iii) $(s, r, \mu) = (5, 5, 1)$,

(iv) $(s, r, \mu) = (7, 7, 1)$.

Proof. We use Theorem 4.3.1.

(i) $p = 3, i = 1, j = 0 \Rightarrow (s, r, \mu) = (3, 3, 1)$ net exists.

(ii) $p = 2, i = 2, j = 0 \Rightarrow (s, r, \mu) = (4, 4, 1)$ net exists.

(iii) $p = 5, i = 1, j = 0 \Rightarrow (s, r, \mu) = (5, 5, 1)$ net exists.

(iv) $p = 7, i = 1, j = 0 \Rightarrow (s, r, \mu) = (7, 7, 1)$ net exists.

Corollary 4.3.5. *Regular thin near octagons with the following parameters do exist:*

(i) $(1, t_2, t_3, t) = (1, 0, 1, 2)$,

(ii) $(1, t_2, t_3, t) = (1, 0, 2, 3)$,

(iii) $(1, t_2, t_3, t) = (1, 0, 3, 4)$,

(iv) $(1, t_2, t_3, t) = (1, 0, 5, 6)$.

Proof. We use Corollary 4.3.4 and Theorem 4.3.3.

Example 4.3.6. We construct a regular thin near octagon with parameters $(1, t_2, t_3, t) = (1, 0, 1, 2)$ from a given corresponding net $(s, r, \mu) = (3, 3, 1)$. From Corollary 4.3.4, we know 9 points and 9 blocks of $(3, 3, 1)$-net.

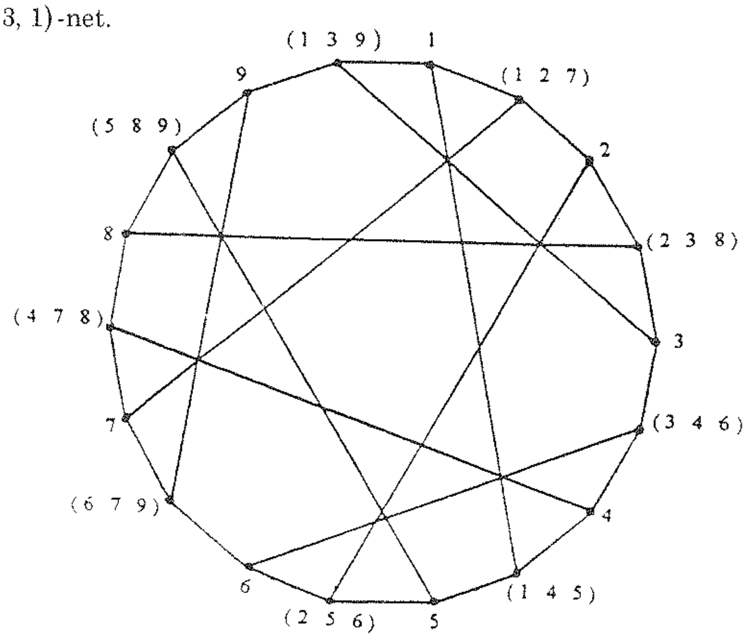

Figure 1

We join each block to the points it contains. We obtain a regular thin near octagon $\mathcal{G} = (\wp, \ell)$ with parameters $(1, 0, 1, 2)$ and for any $x \in \wp$, we have

$$\begin{aligned}|\wp| &= 1 + |\Delta_1(x)| + |\Delta_2(x)| + |\Delta_3(x)| + |\Delta_4(x)| \\ &= 1 + 3 + 6 + 6 + 2 \\ &= 18.\end{aligned}$$

$|\ell|$ = The number of edges (or lines)

$= 27$

(see above graph).

Conjecture 4.3.7. A regular thin near octagon with parameters $(1, t_2, t_3, t) = (1, 0, 4, 5)$ does not exist.

4.4. Parameter sets of the form $(s, t_2, t_3, t) = (1, 1, 2u, 2u + 1)$, $u \geq 1$

In this family of parameters $(s, t_2, t_3, t) = (1, 1, 2u, 2u + 1)$, where $u \geq 1$, we have four feasible parameter sets with $|\wp| \leq 100$ (see Corollary 3.9).

Theorem 4.4.1. *Regular thin near octagon with the following parameters do exist:*

(i) $(1, t_2, t_3, t) = (1, 1, 2, 3)$,

(ii) $(1, t_2, t_3, t) = (1, 1, 4, 5)$,

(iii) $(1, t_2, t_3, t) = (1, 1, 6, 7)$,

(iv) $(1, t_2, t_3, t) = (1, 1, 8, 9)$.

Proof. We use Theorems 4.3.1, 4.3.2, and 4.3.3.

(i) $p = 3, i = 1, j = 1 \Rightarrow (s, r, \mu) = (2, 4, 2)$-net exists.

But this implies that a regular thin near octagon with parameters $(1, 1, 2, 3)$ exists.

(ii) $p = 3, \alpha = 1, \beta = 1 \Rightarrow (s, r, \mu) = (3, 6, 2)$-net exists.

But this implies that a regular thin near octagon with parameters $(1, 1, 4, 5)$ exists.

(iii) $p = 2, i = 2, j = 1 \Rightarrow (s, r, \mu) = (4, 8, 2)$-net exists.

But this implies that a regular thin near octagon with parameters $(1, 1, 6, 7)$ exists.

(iv) $p = 5, \alpha = 1, \beta = 1 \Rightarrow (s, r, \mu) = (5, 10, 2)$-net exists.

But this implies that a regular thin near octagon with parameters $(1, 1, 8, 9)$ exists.

Example 4.4.2. We construct a regular thin near octagon with parameters $(1, 1, 2, 3)$ from the corresponding net $(s, r, \mu) = (2, 4, 2)$. This net has 8 points, namely, $\zeta = \{1, 2, ..., 8\}$ and 8 blocks partitioned into 4 parallel classes of 2 blocks each with every block containing 4 points. If β is the set of blocks, then

$$\beta = \{(1, 5, 6, 8), (2, 3, 4, 7)\} \cup \{(1, 2, 4, 5), (3, 6, 7, 8)\}$$
$$\cup \{(1, 4, 7, 8), (2, 3, 5, 6)\} \cup \{(1, 2, 6, 7), (3, 4, 5, 8)\}.$$

Now, we join each block to the points it contains (see graph below).

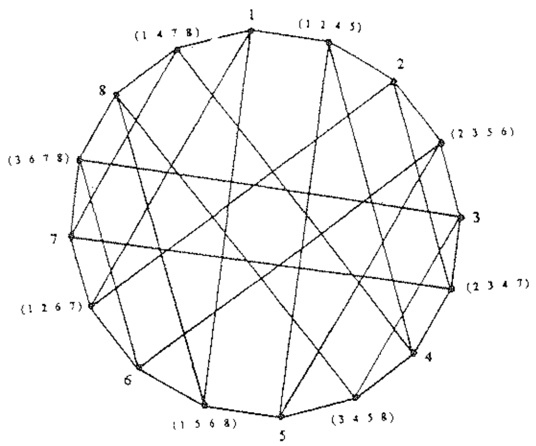

Figure 2

We obtain a regular thin near octagon $\mathcal{G} = (\wp, \ell)$ with parameters $(1, 1, 2, 3)$ and for any $x \in \wp$, we have

$$\wp = \zeta \cup \beta$$

and

$$|\wp| = 1 + |\Delta_1(x)| + |\Delta_2(x)| + |\Delta_3(x)| + |\Delta_4(x)|$$
$$= 1 + 4 + 6 + 4 + 1$$
$$= 16,$$

$$| \ell | = \text{The number of edges (or lines)}$$
$$= 32.$$

4.5. Parameter sets of the form $(s, t_2, t_3, t) = (1, u, 3u + 1, 3u + 2), u \geq 0$

In this family of parameters $(s, t_2, t_3, t) = (1, u, 3u + 1, 3u + 2)$, where $u \geq 0$, we have five feasible parameter sets with $| \wp | \leq 100$ (see Corollaries 3.7 and 3.9).

Theorem 4.5.1. *Regular thin near octagons with the following parameters do exist*:

(i) $(1, t_2, t_3, t) = (1, 0, 1, 2)$,

(ii) $(1, t_2, t_3, t) = (1, 1, 4, 5)$,

(iii) $(1, t_2, t_3, t) = (1, 2, 7, 8)$,

(iv) $(1, t_2, t_3, t) = (1, 3, 10, 11)$.

Proof. (i) Regular thin near octagon with parameters $(1, 0, 1, 2)$ exists. This case has been discussed before (see Example 4.3.6).

(ii) Let $A_1 = \{1, 2, ..., 5, \infty\}$, $A_2 = \{1', 2', ..., 5', \infty'\}$, $A_3 = \{1'', 2'', ..., 5'', \infty''\}$, where $A_i \cap A_j = \varnothing; 1 \leq i \leq 3, 1 \leq j \leq 3, i \neq j$.

Let B be the set of 15 blocks of the resolvable design with parameters $(v, b, r, k, \lambda) = (6, 15, 5, 2, 1)$ given by

$$\{(1\ 2), (3\ 5), (4\ \infty)\} \text{ modulo } 5.$$

Let $\wp = \{\alpha, \alpha', \alpha''\} \cup A_1 \cup A_2 \cup A_3 \cup B$, where $\alpha, \alpha', \alpha''$ are new symbols. Thus $| \wp | = 36$.

We now construct a graph $\mathcal{G}$ on the point set $\wp$ as follows:

(A) α is joined to each point in A_1,

(B) α' is joined to each point in A_2,

(C) α'' is joined to each point in A_3,

(D) $(1\ 2)$ is joined to the points $1, 2; 3', 5'; 4'', \infty''$ modulo 5,

(E) (3 5) is joined to the points 3, 5; 4′, ∞′; 1″, 2″ modulo 5,

(F) (4 ∞) is joined to the points 4, ∞; 1′, 2′; 3″, 5″ modulo 5,

where ∞, ∞′ and ∞″ remain unchanged under all the automorphisms modulo 5.

This construction gives us the graph of a regular thin near octagon with parameters (1, 1, 4, 5) and for any $x \in \wp$,

$$\begin{aligned} |\wp| &= 1 + |\Delta_1(x)| + |\Delta_2(x)| + |\Delta_3(x)| + |\Delta_4(x)| \\ &= 1 + 6 + 15 + 12 + 2 \\ &= 36. \end{aligned}$$

(iii) There exists a resolvable design with parameters $(v, b, r, k, \lambda) = (9, 12, 4, 3, 1)$ given by

$$\{(1\ 6\ 7), (2\ 3\ 5), (4\ 8\ \infty)\} \text{ modulo } 8.$$

From this design we can obtain a multiple design (also resolvable) with parameters $(v, b, r, k, \lambda) = (9, 24, 8, 3, 2)$ by taking all the blocks of above design twice. Then we can construct the graph of a regular thin near octagon with parameters (1, 2, 7, 8) and for any $x \in \wp$,

$$\begin{aligned} |\wp| &= 1 + |\Delta_1(x)| + |\Delta_2(x)| + |\Delta_3(x)| + |\Delta_4(x)| \\ &= 1 + 9 + 24 + 18 + 2 \\ &= 54. \end{aligned}$$

(iv) There exists a resolvable design with parameters $(v, b, r, k, \lambda) = (12, 33, 11, 4, 3)$ given by

$$\{(0, 1, 3, 7), (2, 4, 9, 10), (\infty, 5, 6, 8)\} \text{ modulo } 11.$$

As in part (ii), we can now construct the graph of a regular thin near octagon with parameters (1, 3, 10, 11) and for any $x \in \wp$,

$$\begin{aligned} |\wp| &= 1 + |\Delta_1(x)| + |\Delta_2(x)| + |\Delta_3(x)| + |\Delta_4(x)| \\ &= 1 + 12 + 33 + 24 + 2 \\ &= 72. \end{aligned}$$

Theorem 4.5.2 [11]. *The existence of a regular thin near octagon with parameters* $(s, t_2, t_3, t) = (1, k-1, mk-2, mk-1)$, *implies that the existence of a resolvable design with parameters* $(v, b, r, k, \lambda) = (mk, m(mk-1), mk-1, k, k-1)$.

Corollary 4.5.3. *A regular thin near octagon with parameters* $(s, t_2, t_3, t) = (1, 4, 13, 14)$ *does not exist.*

Proof. Suppose, by way of contradiction, there exists a regular thin near octagon with parameters $(s, t_2, t_3, t) = (1, 4, 13, 14)$. Then the above Theorem 4.5.2 implies that there exists a resolvable design with parameters $(v, b, r, k, \lambda) = (15, 42, 14, 5, 4)$. But we know that a resolvable design with parameters $(v, b, r, k, \lambda) = (15, 42, 14, 5, 4)$ does not exist (see [10]). So we get a contradiction and this completes the proof.

4.6. Parameter sets of the form $(s, t_2, t_3, t) = (1, 1, u, 2u+1)$, $u \geq 2$

In this family of parameters $(s, t_2, t_3, t) = (1, 1, u, 2u+1)$, where $u \geq 2$, we have only two feasible parameter sets with $|\wp| \leq 100$ (see Corollary 3.5).

Theorem 4.6.1. *Regular thin near octagon with the following parameters do not exist:*

(i) $(1, t_2, t_3, t) = (1, 1, 3, 7)$,

(ii) $(1, t_2, t_3, t) = (1, 1, 2, 5)$.

Proof. (i) Suppose, by way of contradiction, there exists a regular thin near octagon with parameters $(1, 1, 3, 7)$.

Let $x \in \wp$. Then $|\Delta_0(x)| = 1$, $|\Delta_1(x)| = 8$, $|\Delta_2(x)| = 28$, $|\Delta_3(x)| = 42$, and $|\Delta_4(x)| = 21$.

Thus $|\wp| = 1 + 8 + 28 + 42 + 21 = 100$.

Let

$$\Delta_1(x) = \{a_1, a_2, \ldots, a_8\},$$

$$\Delta_2(x) = \{x_1, x_2, \ldots, x_{28}\},$$

$$\Delta_3(x) = \{y_1, y_2, ..., y_{24}\},$$

$$\Delta_4(x) = \{z_1, z_2, ..., z_{21}\}.$$

x_1 is adjacent to exactly $1 + t_2 = 2$ points of $\Delta_1(x)$, say, a_1, a_2, and x_1 is adjacent to exactly $t - t_2 = 6$ points of $\Delta_3(x)$, say, $y_1, y_2, ..., y_6$.

Furthermore, a_1 is adjacent to 6 points of $\Delta_2(x)$ other than x_1. Similarly a_2 is adjacent to 6 points of $\Delta_2(x)$ other than x_1. Also, a_1 and a_2 cannot be adjacent to the same point of $\Delta_2(x)$ other than x_1. Since if a_1 and a_2 are both adjacent to same point of $\Delta_2(x)$ other than x_1, then there are at least 3 distinct paths of length 2 between a_1 and a_2 which is impossible as $t_2 = 1$. So let

a_1 be adjacent to $x_2, x_3, ..., x_7$; and

a_2 be adjacent to $x_8, x_9, ..., x_{13}$.

This implies

$$x_2, x_3, ..., x_7 \in \Delta_2(x_1) \tag{1}$$

and

$$x_8, x_9, ..., x_{13} \in \Delta_2(x_1). \tag{2}$$

Now, through each y_i, $i = 1, 2, ..., 6$; exactly $t - t_3 = 7 - 3 = 4$ paths go towards $\Delta_4(x)$, making the total number of paths through $y_1, y_2, ..., y_6$ towards $\Delta_4(x)$ as 24. But $t_2 = 1$ implies no more 2 of these paths can intersect at same $z_j \in \Delta_4(x)$. So there are at least $\frac{24}{2} = 12 z_j$'s, $j = 1, 2, ..., 12$ at distance 2 from x_1. Thus

$$z_1, z_2, ..., z_{12} \in \Delta_2(x_1). \tag{3}$$

Through each y_i, $i = 1, 2, ..., 6$; exactly $1 + t_3 = 4$ paths go towards $\Delta_2(x)$. $d(y_1, a_1) = 2 \Rightarrow$ there are exactly two paths between y_1 and a_1. One of these paths is: $y_1 \sim x_1 \sim a_1$ (where $\sim$ denotes adjacency). Suppose second path is: $y_1 \sim x_2 \sim a_1$.

Similarly $d(y_1, a_2) = 2 \Rightarrow$ there are exactly two paths between y_1 and a_2. One of these paths is: $y_1 \sim x_1 \sim a_2$. Suppose second path is: $y_1 \sim x_8 \sim a_2$.

Thus $y_1 \sim x_1, x_2, x_8$. But y_1 is adjacent to 4 points of $\Delta_2(x)$.

This fourth point cannot belong to the set $\{x_3, x_4, \ldots, x_7\}$, for otherwise we shall have 3 paths between y_1 and a_1.

This fourth point cannot belong to the set $\{x_9, x_{10}, \ldots, x_{13}\}$ either, for otherwise we shall have 3 paths between y_1 and a_2.

Thus we have, say, $y_1 \sim x_{14}$, where $x_{14} \notin \{x_1, x_2, \ldots, x_{13}\}$. Similarly we can prove that $y_2 \sim x_{15}$; $y_3 \sim x_{16}$; $y_4 \sim x_{17}$; where $x_{14}, x_{15}, x_{16}, x_{17}$ are all distinct and

$$\{x_1, x_2, \ldots, x_{13}\} \cap \{x_{14}, x_{15}, x_{16}, x_{17}\} = \varnothing.$$

Therefore

$$x_{14}, x_{15}, x_{16}, x_{17} \in \Delta_2(x_1). \tag{4}$$

We also know from the construction that

$$x \in \Delta_2(x_1). \tag{5}$$

Combining the results (1) to (5), we conclude that

$$\{x, x_2, x_3, \ldots, x_{17}, z_1, z_2, \ldots, z_{12}\} \subseteq \Delta_2(x_1).$$

So $|\Delta_2(x)| \geq 29$. This contradicts that $|\Delta_2(x)| = 28$ for every $x \in \wp$. This completes the proof.

(ii) Proof is much similar to the proof of (i), and is therefore omitted.

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Department of Mathematics
King Abdulaziz University
P. O. Box 80263
Jeddah - 21589, Saudi Arabia